Normal Modes and No Zero Mode Theorem of Scalar Fields in BTZ Black Hole Spacetime

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Abstract

Eigenfunctions for normal modes of scalar fields in BTZ black hole spacetime are studied. Orthonormal relations among them are derived. Quantization for scalar fields is done and particle number, energy and angular momentum are expressed by the creation and annihilation operators. Allowed physical normal mode region is studied on the basis of the no zero mode theorem. Its implication to the statistical mechanics is also studied.

1 Introduction

One of the important aspects of general relativity is black hole physics. Extensive studies in this field have been accomplished from the observational and/or theoretical view points. Many evidences of black holes have been observed including the supermassive black holes at the center of the galaxies [1, 2]. Such black holes are expected to be well described by axisymmetric solutions of the Einstein's field equation. Recently multi-parameter rotating black hole solutions in higher dimensions are studied theoretically from the view point of string theory, M theory, brane world, and with the interest of AdS/CFT correspondence [3, 4, 5].

One of the most interesting aspects of black hole physics is the thermodynamical properties. Black holes can be considered as thermal objects [6, 7]: the entropy is proportional to surface area on the horizon with the Hawking temperature T_{H} [8]. The statistical understanding to the thermodynamics of black holes has been tried in

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the frames of field theories [9], string theory [10] and others. Among them, the brick wall model has been proposed by 't Hoof in order to interpret the area law of the black hole entropy by considering the freedom of scalar fields around black hole horizon in the standard framework of statistical mechanics [11].

On the other hand, there are some problematic issues in understanding the black hole thermoynamics. One of fundamental problems is the super-radiant instability, which occurs in case of rotating black holes. This problem is the case that the flux intensity of scattered outgoing fields to black hole becomes larger than that of ingoing fields under the condition of $\omega - \Omega_{\rm H} m < 0$ [12, 13], where ω and m are frequency and azimuthal angular momentum of field and $\Omega_{\rm H}$ is the angular velocity of black hole. The Boltzmann factor and then the partition function become ill-defined due to the existence of the super-radiance [14]. In BTZ black hole spacetime [15], the super-radiance problem for the scalar fields is also discussed extensively [16, 17, 18, 19]. In our previous papers, statistical mechanics of scalar fields in multi-rotating black hole spacetime are studied [20, 21].

The purpose of this paper is to investigate the normal modes of scalar fields around rotation black holes to define the statistical mechanics well and to understand the super-radiance problem. In order to make the problem clear, we study the BTZ black hole model which is the anti-de Sitter rotating black hole in (2+1) dimension. In case of quasinormal modes in BTZ spacetime, exact analytically treatment have been done [22]. Therefore we expect to obtain the exact eigenfunctions in case of normal modes too. As the boundary conditions, we impose the Dirichlet boundary condition at infinity and the Dirichlet or Neumann boundary condition at horizon for eigenfunctions of normal modes.

The organization of this paper is as follows. In section 2, we prepare notations and definitions of BTZ black hole spacetime and scalar fields for the convenience of the subsequent sections. In section 3, we derive the orthonormal relations among eigenfunctions of normal modes. In section 4, the quantization of the scalar fields will be done. In section 5, particle number, energy and angular momentum are calculated to represent by creation and annihilation operators. In section 6, we will show the theorem of nonexistence of zero normal mode eigenstates. The allowed physical normal mode region will be derived as its application and statistical implication will also be studied in the section too. The results will be summarized in the final section.

2 Scalar fields in BTZ black hole spacetime

This section is the preparation for definitions and notations in the following sections.

The Einstein-Hilbert action with negative cosmological constant ($\Lambda = -1/\ell^2$) in (2+1) dimension is

$$I_G = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R + \frac{2}{\ell^2}) \ . \tag{2.1}$$

In the following, the gravitational constant is normalized as $G = 1/8$D The vacuum

Einstein's equations for this action are

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R - \frac{g_{\mu\nu}}{\ell^2} = 0 \ . \tag{2.2}$$

Banados, Teitelboim and Zanelli (BTZ) solution is in the form as [15],

$$ds^2 = g_{tt}dt^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 \ , \tag{2.3}$$

where the components of metric are

$$g_{tt} = M - \frac{r^2}{\ell^2} \ , \quad g_{t\phi} = -\frac{J}{2} \ , \tag{2.4}$$

$$g_{\phi\phi} = r^2 \ , \quad g_{rr} = (-M + \frac{J^2}{4r^2} + \frac{r^2}{\ell^2})^{-1} \tag{2.5}$$

with black hole mass M and rotation parameter J. The contravariant time component of metric is negative of covariant radial component:

$$g^{tt} = -g_{rr} \ . \tag{2.6}$$

Zeros of their inverse function denote the horizon of black hole

$$r_{\pm}^2 = \frac{M\ell^2}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2\ell^2}}\right) \ , \tag{2.7}$$

where the event horizon is r_+. Note that the BTZ metrics can be rewritten in a diagonal form:

$$ds^2 = \frac{1}{g^{tt}}dt^2 + g_{\phi\phi}(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}}dt)^2 + g_{rr}dr^2 \ . \tag{2.8}$$

The action and the Lagrangian density for the minimally coupled complex scalar field with dimensionless mass μ under the BTZ spacetime is

$$I_{\text{Scalar}} = \int dtdrd\phi\sqrt{-g}\,L_{\text{Scalar}} \ , \tag{2.9}$$

$$L_{\text{Scalar}} = -(g^{\mu\nu}\partial_\mu\Phi^*(x)\partial_\nu\Phi(x) + \frac{\mu}{\ell^2}\Phi^*(x)\Phi(x)) \ . \tag{2.10}$$

The Klein - Gordon equation for this action is

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu) - \frac{\mu}{l^2}\right)\Phi = 0 \ . \tag{2.11}$$

$$\tag{2.12}$$

The background BTZ metric do not depend on time and azimuthal angle variables, and then the scalar field solution is put in the form:

$$\Phi = \mathrm{e}^{-i\omega t}\mathrm{e}^{im\phi}R(r) \ , \tag{2.13}$$

where ω and m ($m = 0, \pm1, \pm2, \cdots$) denote frequency and azimuthal angular momentum. The equation for the radial wave function becomes

$$\left(-g^{tt}(\omega - \frac{J}{2r^2}m)^2 - \frac{m^2}{r^2} + \frac{1}{r}\frac{d}{dr}\frac{r}{g_{rr}}\frac{d}{dr} - \frac{\mu}{\ell^2}\right)R(r) = 0 \ . \tag{2.14}$$

We note that this equation is invariant under the symmetry of (ω, m) and $(-\omega, -m)$.

3 Normal modes and orthonormal relations

We consider the radial wave equation as the eigenvalue equation for the eigenfunction $R_{\omega,m}$ with eigenvalues ω and m:

$$\Delta_r R(r)_{\omega,m} = \left(g^{tt}(\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) R(r)_{\omega,m} \ , \tag{3.1}$$

where Laplacian and the angular velocity at radial position r are defined by:

$$\begin{aligned} \Delta_r &:= \frac{1}{r}\frac{d}{dr}\frac{r}{g_{rr}}\frac{d}{dr} \ , \\ \Omega_r &:= -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{g^{t\phi}}{g^{tt}} = \frac{J}{2r^2} \ . \end{aligned} \tag{3.2}$$

The following two integrations are considered

$$\begin{aligned} &\int dr \sqrt{-g} R^*_{\omega,m} \Delta_r R_{\omega',m} \\ &= \int dr \sqrt{-g} \left(g^{tt}(\omega' - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) R^*_{\omega,m} R_{\omega',m} \ , \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\int dr \sqrt{-g} \Delta_r R^*_{\omega,m} R_{\omega',m} \\ &= \int dr \sqrt{-g} \left(g^{tt}(\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right)^* R^*_{\omega,m} R_{\omega',m} \ . \end{aligned} \tag{3.4}$$

In cases of vanishing the boundary terms, the difference of these integrations becomes

$$\begin{aligned} 0 &= \int dr \sqrt{-g} \left(- \left(g^{tt}(\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right)^* + \left(g^{tt}(\omega' - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) \right) \\ &\times \ R^*_{\omega,m} R_{\omega',m} \\ &= \int dr \sqrt{-g}\, g_{rr} \, (\omega^* - \omega')(\omega^* + \omega' - 2\Omega_r m) R^*_{\omega,m} R_{\omega',m} \ . \end{aligned} \tag{3.5}$$

For the case of $\omega = \omega'$, this relation shows the reality of eigenvalue ω. For the case of $\omega \neq \omega'$, this relation shows the orthonormal relations among eigenfunctions:

$$\int dr \sqrt{-g} \, (-g^{tt}) \, (\omega + \omega' - 2\Omega_r m) R^*_{\omega,m} R_{\omega',m} = \delta_{\omega,\omega'} \ , \tag{3.6}$$

Similarly, we have orthogonal relation for a couple of positive and negative azimuthal angular momentum $(m, -m)$ as

$$\int dr \sqrt{-g} \, g^{tt} \, (\omega - \omega' - 2\Omega_r m) R_{\omega,m} R_{\omega',-m} = 0 \ . \tag{3.7}$$

Defining the full eigenfunction with normalization factor $N_{\omega,m}$

$$f_{\omega,m} := N_{\omega,m} \mathrm{e}^{-i\omega t} \mathrm{e}^{im\phi} R_{\omega,m} \ , \tag{3.8}$$

we have the full orthonormal relations:

$$
\begin{aligned}
\int_\Sigma d\phi dr \sqrt{-g}\,(-g^{tt})\,(\omega+\omega'-2\Omega_r m)\, f^*_{\omega,m} f_{\omega',m'} &= \delta_{\omega,\omega'}\delta_{m,m'} \ , \\
\int_\Sigma d\phi dr \sqrt{-g}\,(-g^{tt})\,(\omega-\omega'-2\Omega_r m)\, f_{\omega,m} f_{\omega',m'} &= 0 \ ,
\end{aligned}
\tag{3.9}
$$

where the integration region Σ denotes $0 \le \phi < 2\pi$ and $r_{\rm H} \le r < \infty$ [1]. For more compact notation, the inner products are introduced:

$$
(A,B) := \int_\Sigma d\phi dr \sqrt{-g}(-ig^{t\nu})\left(A^*(x)\partial_\nu B(x) - \partial_\nu A^*(x) B(x)\right) \ , \tag{3.10}
$$

and the general form of orthonormal relations are obtained:

$$
\begin{aligned}
(f_{\omega,m}, f_{\omega',m'}) &= -(f^*_{\omega,m}, f^*_{\omega',m'}) = \delta_{\omega,\omega'}\delta_{m,m'} \\
(f^*_{\omega,m}, f_{\omega',m'}) &= (f_{\omega,m}, f^*_{\omega',m'}) = 0 \ .
\end{aligned}
\tag{3.11}
$$

It should be noted that the orthonormal relations hold for the allowed normal mode region $0 < \omega - \Omega_r m$.

4 Quantization

We can define the canonical momentum conjugate to scalar field Φ as

$$
\Pi := \frac{\partial L_{\rm Scalar}}{\partial\, \partial_t \Phi} = -g^{t\nu}\partial_\nu \Phi^\dagger = -g^{tt}(\partial_t \Phi^\dagger + \Omega_r \partial_\phi \Phi^\dagger) \ , \tag{4.1}
$$

where $\dagger$ denotes the Hermitian conjugate operation for the quantized fields. We impose the equal time commutation relations among fields and their momenta:

$$
\begin{aligned}
\left[\Phi(t,r,\phi), \Pi(t,r',\phi')\right] &= \frac{i}{\sqrt{-g}}\delta(r-r')\delta(\phi-\phi') \ , \\
\left[\Phi(t,r,\phi)^\dagger, \Pi(t,r',\phi')^\dagger\right] &= \frac{i}{\sqrt{-g}}\delta(r-r')\delta(\phi-\phi') \ ,
\end{aligned}
\tag{4.2}
$$

and others are zero. We make the normal mode expansion for the scalar fields and the conjugate momenta as

$$
\begin{aligned}
\Phi &= \sum_{\omega,m}\left(a_{\omega,m} f_{\omega,m} + b^\dagger_{\omega,m} f^*_{\omega,m}\right) \ , \\
\Pi &= -ig^{tt}\sum_{\omega,m}(\omega - \Omega_r m)(a^\dagger_{\omega,m} f^*_{\omega,m} - b_{\omega,m} f_{\omega,m}) \ .
\end{aligned}
\tag{4.3}
$$

[1] In order to regularize the divergence on the horizon, the cutoff parameter is introduced in explicit construction of normal mode solutions [23]. The cutoff parameter plays the similar role in the brick wall model [11]

Expansion coefficients are inversely expressed by fields and their momenta using the orthonormal relations:

$$
\begin{aligned}
a_{\omega,m} &= (f_{\omega,m}, \Phi) \\
&= \int_\Sigma d\phi dr \sqrt{-g} \Big(i f^*_{t,\omega,m}(t,r,\phi)\Pi^\dagger(t,r,\phi) \\
&\quad - g^{tt}(\omega - \Omega_r m) f^*_{t,\omega,m}(t,r,\phi)\Phi(t,r,\phi) \Big) \ , \\
b^\dagger_{\omega,m} &= -(f^*_{\omega,m}, \Phi) \\
&= -\int_\Sigma d\phi dr \sqrt{-g} \Big(i f_{t,\omega,m}(t,r,\phi)\Pi(t,r,\phi)^\dagger \\
&\quad + g^{tt}(\omega - \Omega_r m) f_{t,\omega,m}(t,r,\phi)\Phi(t,r,\phi) \Big) \ .
\end{aligned} \tag{4.4}
$$

Completeness relations are derived from the consistency of normal mode expansions of fields:

$$
\begin{aligned}
& \sum_{\omega,m} (-g^{tt})(\omega - \Omega_r m) \left(f_{\omega,m}(t,r,\phi) f_{\omega,m}(t,r',\phi')^* + f_{\omega,m}(t,r,\phi)^* f_{\omega,m}(t,r',\phi') \right) \\
= & \ \frac{1}{\sqrt{-g}} \delta(r-r')\delta(\phi-\phi') \ , \\
& \sum_{\omega,m} \left(f_{\omega,m}(t,r,\phi) f_{\omega,m}(t,r',\phi')^* - f_{\omega,m}(t,r,\phi)^* f_{\omega,m}(t,r',\phi') \right) = 0 \ .
\end{aligned} \tag{4.5}
$$

The commutation relations between annihilation and creation operators are derived using the completeness relations:

$$
[a_{\omega,m}, a^\dagger_{\omega',m'}] = \delta_{\omega,\omega'}\delta_{m,m'} \ , [b_{\omega,m}, b^\dagger_{\omega',m'}] = \delta_{\omega,\omega'}\delta_{m,m'} \ , \tag{4.6}
$$

and others are zero.

5 Particle number, energy and angular momentum

¿From the symmetry of the action for the scalar filed, conserved particle number, energy and angular momentum are derived and expressed by the creation and annihilation operators.

1. Particle number

   ¿From the phase translation invariance of the action, particle number current is defined

   $$
   j^\mu = -i g^{\mu\nu}(\Phi^\dagger \partial_\nu \Phi - \partial_\nu \Phi^\dagger \Phi) \ , \tag{5.1}
   $$

   which is shown to satisfy the current conservation

   $$
   j^\mu_{\ ;\mu} = 0 \ . \tag{5.2}
   $$

The corresponding particle number is

$$
\begin{aligned}
N: &= \int_\Sigma d\phi dr \sqrt{-g}\, j^{\,t} \\
&= \int_\Sigma d\phi dr \sqrt{-g}\, (\Pi\Phi - \Phi^\dagger \Pi^\dagger)\ . \qquad (5.3)
\end{aligned}
$$

The particle number is also expressed by the creation and annihilation operators by using normal mode expansion of fields and their momenta

$$
N = \sum_{\omega,m} (a^\dagger_{\omega,m} a_{\omega,m} - b_{\omega,m} b^\dagger_{\omega,m})\ . \qquad (5.4)
$$

2. Energy and Angular Momentum
   Because the metrics don't depend on time and azimuthal angle, two Killing vectors exist

$$
\xi^\mu_{(t)} = (1,0,0) \quad , \quad \xi^\mu_{(\phi)} = (0,0,1)\ . \qquad (5.5)
$$

Defining the energy-momentum tensor

$$
\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta I_{\mathrm{M}}}{\delta g_{\mu\nu}} \\
&= \partial_\mu \Phi^\dagger \partial_\nu \Phi + \partial_\nu \Phi^\dagger \partial_\mu \Phi - g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \Phi^\dagger \partial_\beta \Phi + \frac{\mu}{\ell^2} \Phi^\dagger \Phi)\ , \qquad (5.6)
\end{aligned}
$$

local conservation laws hold for two Killing vectors

$$
(\xi^\mu_{(i)} T^\nu_\mu)_{;\nu} = 0 \quad , \quad for \quad i = t, \phi\ . \qquad (5.7)
$$

Corresponding conservative quantities are energy and angular momentum

$$
\begin{aligned}
E &= -\int_\Sigma d\phi dr \sqrt{-g}\, (\xi^\mu_{(t)} T^t_\mu) \\
&= \int_\Sigma d\phi dr \sqrt{-g} (-g^{tt} \Phi^\dagger_{,t} \Phi_{,t} + g^{\phi\phi} \Phi^\dagger_{,\phi} \Phi_{,\phi} + g^{rr} \Phi^\dagger_{,r} \Phi_{,r}) \qquad (5.8) \\
L &= \int_\Sigma d\phi dr \sqrt{-g}\, (\xi^\mu_{(\phi)} T^t_\mu) \\
&= \int_\Sigma d\phi dr \sqrt{-g} (g^{tt} (\Phi^\dagger_{,t} \Phi_{,\phi} + \Phi^\dagger_{,\phi} \Phi_{,t}) + 2 g^{t\phi} \Phi^\dagger_{,\phi} \Phi_{,\phi})\ . \qquad (5.9)
\end{aligned}
$$

The energy and angular momentum are expressed by the creation and annihilation operators by using normal mode expansion of fields and their momenta

$$
\begin{aligned}
E &= \sum_{\omega,m} \omega (a^\dagger_{\omega,m} a_{\omega,m} + b_{\omega,m} b^\dagger_{\omega,m})\ , \qquad (5.10) \\
L &= \sum_{\omega,m} m (a^\dagger_{\omega,m} a_{\omega,m} + b_{\omega,m} b^\dagger_{\omega,m})\ . \qquad (5.11)
\end{aligned}
$$

The effective energy, which is the energy taking into the rotation effect on the horizon, is expressed as

$$E - \Omega_{\rm H} L \;=\; \sum_{\omega,m} (\omega - \Omega_{\rm H} m)(a^{\dagger}_{\omega,m} a_{\omega,m} + b_{\omega,m} b^{\dagger}_{\omega,m}) \ , \tag{5.12}$$

where $\Omega_{\rm H} = J/2r_+^2$ is the angular velocity on the horizon. The effective energy is positive definite for the allowed normal mode region $0 < \omega - \Omega_{\rm H} m$ except for the zero point energy.

6 No zero mode theorem

In this section, we consider the zero mode eigenstates, which are defined as the stats of $0 = \omega - \Omega_{\rm H} m$ with $-\infty < m < \infty$. We will show that they do not exist and the allowed normal mode region is $0 < \omega - \Omega_{\rm H} m$. We also show that the statistical mechanics for the Hartle-Hawking state is defined well for BTZ black hole spacetime. We impose Dirichlet or Neumann boundary condition at horizon and Dirichlet boundary condition at infinity because spacetime is anti-de Sitter.

Statement 1. Eigenfunction of normal mode for $\omega = m = 0$ does not exist.

Proof General radial eigenfunction of normal mode for $\omega = m = 0$ is obtained solving the radial wave equation in case of massless scalar fields $\mu = 0$ as

$$R_{o,o} = c_1 \ln\left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right) + c_2 \ , \tag{6.1}$$

where c_1, c_2 are integration constants. This solution cannot satisfy both of the boundary conditions at horizon and infinity.

Statement 2. (No zero mode theorem) Eigenfunctions of normal modes for $0 = \omega - \Omega_{\rm H} m$ don't exist.

Proof The radial eigenfunctions of normal mode for $0 = \omega - \Omega_{\rm H} m$ satisfying the boundary condition at infinity is obtained by the hypergeometric function as

$$R_{\omega(=\Omega_{\rm H} m),m} = \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^b \frac{1}{\Gamma(2b)} F\left(b - ic, b + ic, 2b; \frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right) \ , \tag{6.2}$$

where parameters are

$$b = \frac{1 + \sqrt{1 + \mu}}{2} \quad , \quad c = \frac{\ell m}{2r_+} \ . \tag{6.3}$$

This solution also cannot satisfy the boundary condition at horizon. Note that the solution reduces the first term in eq. (6.1) for $\omega = m = \mu = 0$.

Stament 3. The allowed physical normal mode region is $0 < \omega - \Omega_{\rm H} m$ with $-\infty < m < \infty$.

Proof First remark that the allowed physical normal mode region is $0 < \omega$ with $-\infty < m < \infty$ for the case of no rotation $J = 0$. We assume that the normal mode is analytic for rotation parameter J. After switching on the rotation $J \neq 0$, the allowed physical normal mode region shifts from $0 < \omega$ to $0 < \omega - \Omega_{\rm H} m$ because normal modes cannot across each other.

It should be noted that the mode of $0 = \omega - \Omega_{\rm H} m$ is a special mode in the sense that it is the unique solution which satisfies the Dirichlet boundary condition at infinity but deverges at horizon. As a consequence, normal modes are devided into two regions: one region is $0 < \omega - \Omega_{\rm H} m$ with $-\infty < m < \infty$ and the other region is $0 > \omega - \Omega_{\rm H} m$ with $-\infty < m < \infty$ devided by the zero mode. We can confirm this result by the explicit construction of the normal mode eigenfunctions [23].

Statement 4. Statistical mechanics for the scalar fields around BTZ black hole spacetime is well-defined.

Proof The allowed region for the total effective energy becomes $0 < E - \Omega_{\rm H} L$ from the expression in eq. (5.12). Then the Boltzmann factors and then the partition function taking account of rotating effect by Hartle and Hawking [24] become well-defined:

$$Z = \mathrm{Tr}\ \exp(-\beta_{\rm H}(E - \Omega_{\rm H} L))\ , \tag{6.4}$$

where trace is taken for occupation numbers of scalar particles and $\beta_{\rm H}$ denotes the inverse of Hawking temperature [8].

7 Summary

We have investigated normal modes of scalar fields around BTZ black hole spacetime with the Dirichlet or Neumann boundary condition at horizon and convergence to zero at infinity. Orthonormal relations and completeness relations for normal mode eigenfunctions are derived. Using these relations, conserved charge, energy and angular momentum are expressed by creation and annihilation operators of scalar particles. Zero modes of $0 = \omega - \Omega_{\rm H} m$ are shown not to exist and the allowed normal mode region is shown to be $0 < \omega - \Omega_{\rm H} m$. ¿From the allowed normal mode region, the total effective energy should be positive; $0 < E - \Omega_{\rm H} L$, which guarantees the Boltzmann factors for the rotating black holes to be well-defined. This fact shows that the super-radiance does not occur for BTZ black hole spacetime and is consistent with the negative value of the imaginary part for quasinormal frequency [22].

We can extend our result of (2+1) dimensional BTZ black hole spacetime to (3+1) or more higher dimensional rotating black hole spacetime, which will help for the deeper understanding of scalar field entropy around the rotating black holes [25]. We also try to study the relation between our method and the treatment of the conformal field theory, which is developed extensively in view of exact AdS/CFT correspondence in BTZ spacetime [26, 27, 28, 29].

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